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WEIERSTRASS INTEGRABILITY IN LIÉNARD DIFFERENTIAL SYSTEMS

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ABSTRACT. In this work we study the Liénard differential systems that admit a Weierstrass first integral or a Weierstrass inverse integrating factor.

1. Introduction

We consider the Liénard differential equations of the form

$$\ddot{x} + f(x)\dot{x} + g(x) = 0, \tag{1}$$

where $f, g : \mathbb{R} \to \mathbb{R}$ are real C^k–functions and the dot denotes, as usual, derivative with respect to the independent variable t. Here k runs over $1, 2, \ldots, \infty, \omega$. Of course C^ω denotes the class of analytic functions. In what follows we denote by F and G the two functions such that $F'(x) = f(x)$, $G'(x) = g(x)$, $F(0) = 0$ and $G(0) = 0$. We can write the differential equation (1), taking coordinates x and $Y = \dot{x}$, as the differential system

$$\dot{x} = Y, \qquad \dot{Y} = -f(x)Y - g(x), \tag{2}$$

and in the coordinates x and $y = \dot{x} + F(x)$, it becomes

$$\dot{x} = y - F(x), \qquad \dot{y} = -g(x). \tag{3}$$

The applications of these differential equations to the natural sciences and technology is enormous, and well justifies their continued study. Moreover as we will see many other systems have transformations which bring them to the form (1) or to the related systems (2) and (3). The most important recent lines of research in the Liénard differential systems are the study of the center problem and of their limit cycles, see [6, 7, 12, 19, 20] and references therein. In the present paper we study the integrability problem for such systems.

For the moment a universal definition of integrability for a dynamical system seems elusive. There exist several definitions of integrability and it is still an open problem to clarify completely the relationships between them, see [15]. For instance, system (2) can be written as the first–order ordinary differential equation

$$\frac{dY}{dx} = -f(x) - \frac{g(x)}{Y}. \tag{4}$$

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and by Proposition 1 given in [10] the Vessiot–Guldbert–Lie algebra associated to the Liénard differential equation (4) has infinite dimension, and consequently this equation cannot be written as a Riccati differential equation using a change of variables of the form $(x, y) \mapsto (x, z = \phi(Y))$. Hence in the sense of the Lie group analysis (see [1, 25, 16]) the Liénard differential equation (4) is not integrable. In [10] special coordinate changes of the form $(x, y) \mapsto (x, \Phi(x, Y))$ are used to enlarge a bit the class of Liénard differential equations which can be transformed into a Riccati differential equation.

From now on we concentrate our attention in the Liénard differential system of the form (3). Let U the domain of definition of system (3). Let φ be the C^k flow defined by the differential system (3). We denote by Σ the union of all separatrices of φ, for a definition of separatrix, see [3, 22]. It is known that Σ is a closed invariant subset of U. A component C of $U \setminus \Sigma$ with the restricted flow $\varphi|_C$, is called a *canonical region* of φ. Then the local flow $\varphi|_C$ has a C^k first integral H^C on every canonical region C of φ, see [3, 18]; i.e. $(y - F(x))H_x^C - g(x)H_y^C|_C = 0$.

When the differential system (3) has a function $H : U \setminus \Sigma \to \mathbb{R}$. which is the same on all canonical regions (i.e. $H|_C = H^C$ for every canonical region C), we say that we have a *first integral* of system (3). A planar differential system having a first integral is called *integrable*.

The objective of the present paper is to study the integrability problem of the Liénard differential equation (1) or of the equivalent systems (2) and (3). Since for such systems the notion of integrability is based on the existence of a first integral the following natural question arises: *Given a differential system* (3) *how to recognize if this differential system has a first integral?, and how to compute it when it exists?*

Currently we cannot yet give an answer to that question and we focus our studies in investigate when the differential system (3) has a Weierstrass first integral or a Weierstrass inverse integrating factor (see the definitions below).

As usual we define $\mathbb{C}[[x]]$ the set of the formal power series in the variable x with coefficients in $\mathbb{C}$ and $\mathbb{C}[y]$ the set of the polynomials in the variable y with coefficients in $\mathbb{C}$. A polynomial of the form

$$\sum_{i=0}^{n} a_i(x)y^i \in \mathbb{C}[[x]][y],$$

is called a *formal Weierstrass polynomial* in y of degree n if and only if $a_n(x) = 1$ and $a_i(0) = 0$ for $i < n$. A formal Weierstrass polynomial whose coefficients are convergent is called *Weierstrass polynomial*, see [2]. We say that a differential system (3) is *Weierstrass integrable* if system (3) admits a first integral or an inverse integrating factor which is a Weierstrass polynomial. In [14] is given this definition in a more general context.

The main objective of this paper is to obtain Liénard differential systems that have Weierstrass first integral or a Weierstrass inverse integrating factor . More precisely: *How to recognize the functions $f(x)$ and $g(x)$ for which the second–order ordinary Liénard differential equation* (1) *or the equivalent differential system* (3) *is Weierstrass integrable?* This is also a difficult problem, and we only will be able to provide a partial answer.

The main results on the existence of Weierstrass first integrals are stated in section 3, see Theorem 4 and Proposition 5; the ones for the existence of Weierstrass inverse integrating factors are given in section 4, see Theorem 8 and Proposition 9. In section 5 there are extensions of Theorems 4 and 8 to a generalized class of Liénard differential systems, see Theorems 10 and 11. Finally two examples of how use our result for finding Weierstrass integrable Liénard differential systems are given in section 6. In section 2 we see that there exist many differential systems that can be transformed to Liénard differential systems.

2. Differential systems that can be transformed to Liénard systems

In this section we will see that several kind of planar differential systems can be reduced to study Liénard differential equations.

Proposition 1 (see [5] and [26] pp. 356–357)**.** *A planar differential system of the form*

$$\begin{aligned} \dot{x} &= f_0(x) - f_1(x)y, \\ \dot{y} &= g_0(x) + g_1(x)y + g_2(x)y^2, \end{aligned}$$

with $f_1(x) \neq 0$*, can be transformed into a Liénard differential system* (2) *doing the change of variables* $Y = (f_0(x) - f_1(x)y) \exp\left(\int_0^x (g_2(\tau) - f_1'(\tau))/f_1(\tau)d\tau\right)$.

One important consequence of Proposition 1 is that the study of the limit cycles of the quadratic systems can be reduced to study the limit cycles of a Liénard differential equation, see for instance [11]. Proposition 1 can be generalized to the following differential systems.

Proposition 2 ([11])**.** *A planar differential system of the form*

$$\begin{aligned} \dot{x} &= f_0(x) - f_1(x)y^n, \\ \dot{y} &= g_1(x)y + g_2(x)y^{n+1}, \end{aligned}$$

with $f_1(x) \neq 0$*, can be transformed into the Liénard system* (2) *doing the change of variables* $Y = (f_0(x) - f_1(x)y^n)\exp(\int_0^x (ng_2(\tau) - f_1'(\tau))/f_1(\tau)d\tau)$.

Finally the following proposition gives some differential systems (which contain the previous ones) that can be transformed to a generalized Liénard system its proof is straightforward.

Proposition 3. *A planar differential system of the form*

$$\begin{aligned} \dot{x} &= f_0(x) - f_1(x)y^n, \\ \dot{y} &= g_0(x) + g_1(x)y + g_2(x)y^{n+1}, \end{aligned} \tag{5}$$

with $f_1(x) \neq 0$*, can be transformed into the generalized Liénard system*

$$\begin{aligned} \dot{x} &= f_0(x)h(x) + Y^n, \\ \dot{Y} &= -f(x)Y - g(x), \end{aligned}$$

doing the change of variables $Y = y \exp\left(\int_0^x g_2(\tau)/f_1(\tau)d\tau\right)$.

In Proposition 3 if we take $n = 1$ and $f_0(x) = 0$ in system (5), then it is transformed to the Liénard differential system (2). For this particular case the authors of [7] studied the center problem.

3. Liénard systems with Weierstrass first integral

In this section we study the Liénard differential system (3) having a Weierstrass first integral.

Theorem 4. *The Liénard differential system* (3) *admits a Weierstrass first integral of the form*

$$H = H_s(x)y^s + H_{s-1}(x)y^{s-1} + \ldots + H_1(x)y + H_0(x), \tag{6}$$

if the functions $H_i(x)$ for $i = 0, 1, \ldots, s$ satisfy

$$\begin{aligned} H_s'(x) = &\ 0, \\ H_{s-1}'(x) = &\ 0, \\ H_{s-i}'(x) - F(x)H_{s-i+1}'(x) - (s-i+2)g(x)H_{s-i+2}(x) = &\ 0, \quad \textit{for } i = 2, \ldots s-1, \\ F(x)H_0'(x) + g(x)H_1(x) = &\ 0. \end{aligned}$$

Moreover a solution of the previous first four differential equations is

$$\begin{aligned} H_s(x) = &\ 1, \\ H_{s-1}(x) = &\ c_{s-1}, \\ H_{s-2}(x) = &\ s\,G(x) + c_{s-2}, \\ H_{s-3}(x) = &\ \int_0^x [(s-1)c_{s-1}g(\tau) + sF(\tau)g(\tau)]d\tau + c_{s-3}, \end{aligned}$$

where c_{s-1}, c_{s-2} and c_{s-3} are arbitrary constants.

Proof. Imposing that system (3) has the first integral (6) we obtain a polynomial in y whose coefficients must be null. Hence, we obtain a recursive differential system to determine the functions $H_i(x)$ for $i = 0, \ldots, s$. The highest power is y^{s+1} and its coefficient is $H_s'(x) = 0$, therefore we can take $H_s(x) = 1$. The next power of y is y^s and its coefficient is $H_{s-1}'(x) = 0$, hence we take $H_{s-1}(x) = c_{s-1}$ an arbitrary constant. Substituting these values in the next equation we obtain $H_{s-2}'(x) - sg(x) = 0$ which implies $H_{s-2}(x) = sG(x) + c_{s-2}$, where c_{s-2} is an arbitrary constant. Finally, the next equation is $H_{s-3}'(x) - (s-1)c_{s-1}g(x) - sF(x)g(x) = 0$ whose solution is given in the statement of the theorem. □

Proposition 5. *The Liénard differential system* (3) *admits a Weierstrass first integral of the form* (6)

(a) *with $s = 1$ if and only if $g(x) = 0$,*
(b) *with $s = 2$ if and only if $g(x) = 0$ or $c_1 + 2F(x) = 0$ where c_1 is an arbitrary constant.*
(c) *with $s = 3$ if and only if $g(x) = 0$ or $c_1 + 2c_2 + 3F(x)^2 + 3G(x) = 0$, where c_1 and c_2 are arbitrary constants.*
(d) *with $s = 4$ if and only if $g(x) = 0$ or*

$$c_1 + 2c_2F(x) + 3c_3F(x)^2 + 4F(x)^3 + 8F(x)G(x) + 3c_3G(x) + 4\int_0^x F(\tau)g(\tau)d\tau = 0,$$

where c_1, c_2, c_3 are arbitrary constants.

Proof. Following the same arguments than in the proof of Theorem 4 the last equation in each case give the compatibility condition given in each statement of the proposition. The arbitrary constants c_i are the arbitrary constants of integration of each function H_i. □

Now we compare the results obtained on the integrability of the Liénard differential systems in Proposition 5 with the results on integrability of the Liénard differential systems having a center. A center is *nondegenerate* if their eigenvalues are purely imaginary.

The next result is due to Poincaré [23, 24] and Liapunov [17], see also Moussu [21].

Theorem 6 (Analytic nondegerate center theorem)**.** *The analytic differential system* (3) *has a nondegerate center at the origin if and only if there exists an analytic first integral defined in a neighborhood of the origin.*

We have the following nice characterization of the nondegenerate centers at the origin for the Liénard differential systems (3), see for instance [5, 6].

Theorem 7 (Center Theorem for analytic Liénard differential systems)**.** *Let* $f(x)$ *and* $g(x)$ *be analytic functions defined in a neighborhood of zero. Then the Liénard differential system* (3) *has a nondegenerate center at the origin if and only if* $f(0) = 0$*,* $g'(0) > 0$ *and* $F(x) = \Phi(G(x))$ *for some analytic function* $\Phi(x)$ *such that* $\Phi(0) = 0$.

Putting together Theorems 6 and 7 we see that a Liénard differential system (3) with $f(x)$ and $g(x)$ analytic functions, $f(0) = 0$ and $g'(0) > 0$ has a local analytic first integral at the origin if and only if $F(x) = \Phi(G(x))$ for some analytic function $\Phi(x)$ such that $\Phi(0) = 0$.

We note that when we have a Weierstrass first integral for the Liénard differential systems (3) (see Proposition 5) we also have relations between the functions $f(x)$ and $g(x)$, but note that in the relationships of Proposition 5 the functions that appear there do not need to be analytic.

4. Liénard systems with Weierstrass inverse integrating factor

In this section we study the Liénard differential system (3) having a Weierstrass inverse integrating factor.

Theorem 8. *The Liénard differential system* (3) *admits a Weierstrass inverse integrating factor of the form*

$$V = V_s(x)y^s + V_{s-1}(x)y^{s-1} + \ldots + V_1(x)y + V_0(x), \tag{7}$$

if the functions $V_i(x)$ *for* $i = 0, 1, \ldots, s$ *satisfy*

$$\begin{aligned} V_s'(x) = &\ 0,\\ V_{s-1}'(x) + V_s(x)F'(x) - F(x)V_s'(x) = &\ 0,\\ V_{s-i}'(x) + V_{s-i+1}(x)F'(x) - F(x)V_{s-i+1}'(x) - (s-i+2)g(x)V_{s-i+2}(x) = &\ 0,\\ \text{for } i = 2, \ldots s-1, &\\ V_0(x)F'(x) - F(x)V_0'(x) - g(x)V_1(x) = &\ 0. \end{aligned}$$

Moreover a solution of the previous first four differential equations is

$$\begin{aligned} V_s(x) = &\ 1,\\ V_{s-1}(x) = &\ -F(x) + c_{s-1},\\ V_{s-2}(x) = &\ s\,G(x) - c_{s-1}F(x) + c_{s-2},\\ V_{s-3}(x) = &\ \textstyle\int_0^x F(\tau)g(\tau)d\tau - s\int_0^x F'(\tau)G(\tau)d\tau + (s-1)c_{s-1}G(x) - c_{s-2}F(x) + c_{s-3}, \end{aligned}$$

where c_{s-1}*,* c_{s-2} *and* c_{s-3} *are arbitrary constants.*

Proof. Imposing that system (3) has the inverse integrating factor of the form (6), as in the proof of Theorem 4, we obtain a polynomial in y whose coefficients must be null. Hence, we obtain a recursive differential system to determine the functions $V_i(x)$ for $i = 0, \dots, s$. The highest degree in y is given by y^{s+1} its coefficient is $V_s'(x) = 0$. Therefore we also can take $V_s(x) = 1$. Substituting this value in the next equation we obtain $V_{s-1}'(x) + F'(x) = 0$, hence we take $V_{s-1}(x) = -F(x) + c_{s-1}$, where c_{s-1} is an arbitrary constant. Substituting these values in the next equation we obtain $V_{s-2}'(x) - sg(x) + c_{s-1}F'(x) = 0$ which implies $V_{s-2}(x) = s\,G(x) - c_{s-1}F(x) + c_{s-2}$, where c_{s-2} is another arbitrary constant. Finally, the next equation is $V_{s-3}'(x) - F(x)g(x) + sF'(x)G(x) - (s-1)c_{s-1}g(x) + c_{s-2}F'(x) = 0$ whose solution is given in the statement of the theorem. □

Proposition 9. *The Liénard differential system* (3) *admits a Weierstrass inverse integrating factor of the form* (7)

(a) *with $s = 1$ if and only if $g(x) + c_0F'(x) = 0$ where c_0 is an arbitrary constant.*

(b) *with $s = 2$ if and only if $c_1g(x) - F(x)g(x) + c_0F'(x) + 2G(x)F'(x) = 0$ where c_0 and c_1 are arbitrary constants.*

(c) *with $s = 3$ if and only if*

$$\begin{gathered} F(x)^2g(x) + 3g(x)G(x) - 4F'(x)\int_0^x F(\tau)g(\tau)d\tau \\ +c_1g(x) + c_2F(x)g(x) - c_0F'(x) - 2c_2G(x)F'(x) = 0, \end{gathered}$$

where c_0, c_1 and c_2 are arbitrary constants.

(d) *with $s = 4$ if and only if*

$$\begin{gathered} F(x)^3g(x) + 4F(x)g(x)G(x) - 4G(x)F'(x) \\ +5(F(x)F'(x) - g(x))\int_0^x F(\tau)g(\tau)d\tau + F'(x)\int_0^x F(\tau)^2g(\tau)d\tau \\ -5F'(x)\int_0^x F'(\tau)\left(\int_0^x F(\tau)g(\tau)\,d\tau\right)d\tau + c_1g(x) + c_2F(x)g(x) + c_3F(x)^2g(x) \\ +3c_3g(x)G(x) - c_0F'(x) - 2c_2G(x)F'(x) - 3c_3F(x)G(x)F'(x) \\ -c_3F'(x)\int_0^x F(\tau)G(\tau)d\tau + 3c_3F'(x)\int_0^x G(\tau)F'(\tau)d\tau = 0, \end{gathered}$$

where c_0, c_1, c_2 and c_3 are arbitrary constants.

Proof. Using the same arguments than in the proof of Theorem 8 the last equation in each case give the compatibility condition given in each statement of the proposition. The arbitrary constants c_i are the arbitrary constants of integration of each function V_i. □

5. Generalized Liénard systems with Weierstrass integrability

In this section we study the generalization of the Liénard differential system (3) given by

$$\dot{x} = \varphi(y) - F(x), \qquad \dot{y} = -g(x), \tag{8}$$

where $\varphi(y)$ is a polynomial of degree k, i.e. $\varphi(y) = y^k + p_{k-1}y^{k-1} + \cdots + p_0$. For system (8) we are going to study the existence of a Weierstrass first integral and the existence of a Weierstrass inverse integrating factor.

Theorem 10. *The generalized Liénard differential system* (8) *with $g(x) \neq 0$ admits a Weierstrass first integral of the form* (6) *if and only if $s > k$. The case $g(x) = 0$ is trivial because system* (8) *has $H = y$ as a first integral.*

Proof. Imposing that system (8) has the first integral (6) we obtain a polynomial in y whose coefficients must be null. Hence, we obtain a recursive system of differential equations for determining the functions $H_i(x)$ for $i = 0, \ldots, s$. The highest power in y is y^{k+s} and its coefficient is $H_s'(x) = 0$. Therefore $H_s(x) =$ constant that we can take equal 1, i.e. $H_s(x) = 1$. The next power of y is y^{k+s-1} and its coefficient is $H_{s-1}'(x) = 0$. The number of equations is $k+s+1$ and the number of functions to be determined is $s+1$ with s arbitrary constants of integration. Hence we have $k+s+1$ equations with $2s+1$ unknowns. Therefore the system will be compatible if $k+s+1 = 2s+1$ that is $k \le s$. Now we assume that $k = s$. The highest power of y is now y^{2s}. With the first $s+1$ equations we determine $H_s, H_{s-1}, \ldots, H_0$ which take the values $H_s = 1$ and $H_{s-i} = c_{s-i}$ for $i = 1, \ldots, s$, where c_{s-i} are arbitrary constants. The next power of y is y^{s-1} which has the following coefficient

$$F(x)H_{s-1}'(x) - sg(x)H_s(x) = 0. \tag{9}$$

Taking into account that $H_{s-1}'(x) = 0$ and $H_s(x) \neq 0$, expression (9) implies $g(x) = 0$. ☐

Theorem 11. *The generalized Liénard differential system* (8) *with* $g(x) \neq 0$ *admits a Weierstrass inverse integrating factor of the form* (7) *if and only if* $s \ge k$. *Moreover if* $k = s$ *the existence of the mentioned inverse integrating factor needs that* $g(x) = (p_{s-1} - c_{s-1})F'(x)/s$, *where* c_{s-1} *is an arbitrary constant.*

Proof. Imposing that system (8) has the inverse integrating factor (7) we obtain, as in the proof of Theorem 10, a polynomial in y whose coefficients must be null. Hence we get a recursive system of differential equations for determining the functions $V_i(x)$ for $i = 0, \ldots, s$. The highest power in y is y^{k+s} and its coefficient is $V_s'(x) = 0$. Therefore we can take $V_s(x) = 1$. The next power of y is y^{k+s-1} and its coefficient is $V_{s-1}'(x) = 0$. The number of equations is $k+s+1$ and the number of functions to be determined is $s+1$ with s arbitrary constants of integration. Hence we have $k+s+1$ equations with $2s+1$ unknowns. Therefore the system will be compatible if $k+s+1 = 2s+1$ that is $k \le s$. Now we assume that $k = s$. The highest power of y is now y^{2s}. With the first s equations we determine $V_s, V_{s-1}, \ldots, V_1$ which take the values $V_s = 1$ and $V_{s-i} = c_{s-i}$ for $i = 1, \ldots, s-1$, where c_{s-i} are arbitrary constants. The power y^s gives the differential equation $V_0'(x) = F'(x)$. The next power of y is y^{s-1} which has the following coefficient

$$F(x)V_{s-1}'(x) - sg(x)V_s(x) - V_{s-1}F'(x) + p_{k-1}V_0'(x) = 0. \tag{10}$$

Taking into account that $V_{s-1}'(x) = 0$, $V_s(x) = 1$ and $V_0'(x) = F'(x)$, from equation (10) we can isolated $g(x) = (p_{s-1} - c_{s-1})F'(x)/s$ where we have take $V_{s-1} = c_{s-1}$ an arbitrary constant. ☐

The following proposition shows the type of results that one can obtain using Theorem 11.

Proposition 12. *The Liénard differential system*

$$\dot{x} = y^k - F(x), \qquad \dot{y} = -g(x), \tag{11}$$

with $F(x) = c\,G(x)^{\frac{k}{k+1}}$ *admits a Weierstrass inverse integrating factor of the form*

$$V = y^{k+1} - c\,G(x)^{\frac{k}{k+1}}\, y + k\,G(x), \tag{12}$$

where c *is an arbitrary constant.*

Proof. From the recursive system of differential equations for the functions V_i with $i = 0, \dots, k+1$. We obtain $V_k = 1$, $V_i = 0$ for $i = 2, \dots, k$, $V_1 = -F(x)$ and $V_0 = kG(x)$. The last equation gives the compatibility condition $F(x) = c\, G(x)^{\frac{k}{k+1}}$, where c is an arbitrary constant. Hence straightforward computations show that the inverse integrating factor (12) satisfies the equation

$$\frac{\partial V}{\partial x}\,\dot{x} + \frac{\partial V}{\partial y}\,\dot{y} - V\left(\frac{\partial \dot{x}}{\partial x} + \frac{\partial \dot{y}}{\partial y}\right) = 0.$$

□

Generalized Liénard systems of the form (11) have been studied in [12] when φ, F and g are analytic functions. For these system it is studied the center problem and the following result is established.

Theorem 13 (Center Theorem for generalized Liénard differential systems)**.** *Consider system* (11) *where all the involved function φ, F and g are analytic functions satisfying $\varphi(y) = y^{2m-1} + \mathcal{O}(y^{2m})$, $F(x) = a_k x^k + \mathcal{O}(x^{k+1})$ and $G(x) = \int_0^x g(\tau)d\tau = x^{2\ell}/2\ell + \mathcal{O}(x^{2\ell+1})$, with m, k and ℓ being positive integers. Assume that $k > \ell(2m-1)/m$. Then system* (11) *has a center at the origin if and only if there exists an analytic function $\Phi(x)$, such that $\Phi(0) = 0$ and satisfying $F(x) = \Phi(G^{1/\ell}(x))$.*

We note that when we have a Weierstrass integrable generalized Liénard differential systems (8) (see Theorems 10 and 11) we also have a relation between the functions $f(x)$ and $g(x)$, but note that in the relationships of Proposition 5 the functions that appear there do not need to be analytic.

6. Examples

Example 1. We consider the following system

$$\dot{x} = y + x^2\ , \qquad \dot{y} = -\,x^3\ , \tag{13}$$

System (13) has a nilpotent center at the origin because the origin is a monodromic singular point and the system is time–reversible due to it is invariant by the symmetry $(x, y, t) \to (-x, y, -t)$. However it has neither a local analytic first integral defined at the origin, nor a formal first integral, see the proof in [4]. System (13) was also studied in [8] from the classical Lie theory point of view showing that it has a generalized nonlinear superposition principle, see also [9]. In [13] using that this nilpotent center is limit of linear type centers, an analytic first integral for this system defined in $\mathbb{R}^2 \setminus \{(0,0)\}$ is computed. System (13) is a Liénard system written in the form (3) where $F(x) = x^2$ and $g(x) = x^3$. Moreover we are going to see that system (13) verifies the statement (b) of Proposition 9. The resolution of the differential equation given in the statement (b) of Proposition 9 is $F(x) = -c_1 + c\sqrt{c_0 + 2G(x)}$ where c_0, c_1 and c are arbitrary constants. Hence system (13) verifies the statement (b) of Proposition 9 with $c_0 = c_1 = 0$ and $c = -\sqrt{2}$ because $G(x) = x^4/4$ and $F(x) = -2\sqrt{x^4/4} = -2\sqrt{G(x)}$. Therefore, system (13) has a Weierstrass inverse integrating factor of the form (7) with $s = 2$. Straightforward computations give that this inverse integrating factor, up to a constant factor, is $V = y^2 + x^2 y + x^4/2$.

Example 2. We consider the polynomial differential system

$$\dot{x} = y - a\left(-x + \frac{x^3}{3}\right), \qquad \dot{y} = -x - \frac{a^2x^3(x^2-4)}{16}, \tag{14}$$

with $a \neq 0$. System (14) was studied by Wilson in 1964 and is the first example of a Liénard system with an algebraic limit cycle. More precisely system (14) has the invariant algebraic curve

$$f_0(x,y,a) := -4 + x^2 + \frac{a^2x^6}{144} - \frac{ax^3y}{6} + y^2 = 0$$

as a limit cycle when $0 < |a| < 2$. For $|a| \geq 2$, the invariant algebraic curve f turns out to contain a singular point, and so it cannot be a limit cycle. Moreover system (14) has two additional invariant algebraic curves

$$f_1, f_2 := y - ax\left(\frac{x^2-6}{12}\right) \pm \frac{\sqrt{a^2-4}}{2}\, x\,,$$

which allow to construct a Darboux first integral, see [8]. System (14) is also a Liénard system written in the form (3) where $F(x) = a(-x + x^3/3)$ and $g(x) = x + a^2x^3(x^2-4)/16$. Moreover it is easy to see that system (14) verifies the statement (c) of Proposition 9 with $c_0 = c_1 = c_3 = 0$ and $c_2 = -4$. Therefore, system (13) has a Weierstrass inverse integrating factor of the form (7) with $s = 3$. Straightforward computations give that this inverse integrating factor, up to a constant factor, is $V = f_0f_1f_2$.

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